Appl. Math. Lett. Vol. 5, No. 4, pp. 91–94, 1992
Printed in Great Britain
0893-9659/92 $5.00 + 0.00
Pergamon Press Ltd

INVERSION OF INCOMPLETE CONE-BEAM DATA

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(Received April 1992)

Abstract—Formulas for the analytical inversion of the incomplete cone-beam data are obtained. These formulas have computational advantages: they involve two one-dimensional integrations over compact regions for the calculation of the Radon transform data or the Fourier transform data and some analytical inversion formulas for incomplete Radon transform or Fourier transform data.

1. INTRODUCTION

Let $f(x) \in C^1(\mathbf{R}^n)$, $n \geq 2$, $f(x) = 0$ for $|x| \geq a$, $D = \operatorname{supp} f$, supp = support, L be a C^1-curve not intersecting D. The cone-beam data are defined to be

$$g(x, \alpha) := \int_0^\infty f(x + t\alpha)\,dt, \quad x \in L, \ \alpha \in S^{n-1}. \tag{1}$$

They are extended from $L \times S^{n-1}$ to $L \times \mathbf{R}^n$ by Formula (1), one has $g(x, \lambda\alpha) = \lambda^{-1} g(x, \alpha)$ for $\lambda > 0$. Let $\hat{f}(p, \alpha) = \int_{\mathbf{R}^3} f(x)\,\delta(p - \alpha \cdot x)\,dx$ be the Radon transform of $f(x)$, δ is the delta-function.

The problem we study is: what are the conditions on L under which the data (1) determine $f(x)$ uniquely, and what are the inversion formulas that allow one to calculate $f(x)$?

There are several papers ([1–6]) which deal to some extent with the above problem. Note that [5, Formula (1.6)] is not clear: it involves integration with respect to p over $\mathbf{R}$ of a function which is not defined outside a finite interval. Our results are new and more convenient from the computational point of view. In Section 2, we state the results, in Section 3 proofs are given.

2. RESULTS

We state the results for $n = 3$. The simpler case $n = 2$ is discussed at the end of this section.

Let $l_\beta := \{y : \ y = t\beta, \ t \in \mathbf{R}, \ \beta \in S^2\}$ be a straight line, π_β be the orthogonal projection onto l_β, $\pi_\beta(D) = [a(\beta), b(\beta)]$, $a(\beta) < b(\beta)$. Let $D_\epsilon := \{x : \ \rho(x, D) < \epsilon\}$, $\rho(x, D)$ is the distance from x to D. *Our basic assumption is:*

$$\text{there exists } \beta \in S^2 \text{ and a neighborhood } D_\epsilon \text{ of } D \text{ such that } \pi_\beta(D_\epsilon) \subset \pi_\beta(L). \tag{2}$$

THEOREM 1. *If (2) holds, then the data $g(x, \alpha)$, $x \in L$, $\alpha \in \omega$, where ω is a neighborhood of β in S^2, however small, determine $f(x)$ uniquely.*

A.G.R. thanks ONR, NSF, USIEF and the Technion for support. This paper was written when A.G.R. was Fulbright Research Professor at the Technion. The research of A.I.Z. was supported in part by a grant from the Ministry of Science and the "Ma-agara"—special project for absorption of the immigrants, in the Department of Mathematics, Technion.

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THEOREM 2. *Under the assumptions of Theorem 1, one has*

$$\hat{f}(p,\alpha)=\begin{cases}0, & \text{for } p<a(\alpha) \text{ or } p>b(\alpha), \quad \alpha\in\omega,\\ \int_{a(\alpha)}^{p} d\lambda \int\limits_{S^2\cap\alpha^\perp} d\eta\, \mathcal{D}_\alpha\, g(B(\lambda,\alpha),\eta), & a(\alpha)\le p\le b(\beta),\ \alpha\in\omega.\end{cases} \tag{3}$$

Here $B(\lambda,\alpha)\in L$ is an arbitrary point of L such that

$$\pi_\alpha(B(\lambda,\alpha))=\lambda\alpha, \quad a(\alpha)\le\lambda\le b(\alpha), \tag{4}$$

$\alpha^\perp$ is the plane orthogonal to α and containing the origin, $d\eta$ is the length element of the circle $S^2\cap\alpha^\perp$, and $\mathcal{D}_\alpha$ is the differentiation of a function defined in a neighborhood of $S^2\cap\alpha^\perp$ in S^2 in the direction α. For example, when $\alpha=(0,0,1)$, one has

$$\mathcal{D}_\alpha\, g(x,\eta):=\left.\frac{\partial g}{\partial\eta_3}\right|_{|\eta|=1,\eta\cdot\alpha=0}=-\left.\frac{\partial g_1}{\partial\theta}\right|_{\theta=\frac{\pi}{2}}, \tag{5}$$

where $g_1:=g_1(x,\theta,\phi):=g(x,\eta)|_{|\eta|=1}$, $\eta_3=\cos\theta$, $\eta_2=\sin\theta\sin\phi$, $\eta_1=\sin\theta\cos\phi$.

Let K be the cone $\mathbf{R}\times\omega$, $\omega_-:=-\omega$. For $\xi\in\mathbf{R}^3\setminus\{0\}$ set $\xi^0=\xi/|\xi|$, $\tilde{f}(\xi)=\mathcal{F}f:=\int\limits_{\mathbf{R}^3}\exp(i\xi x)\,f(x)\,dx$.

THEOREM 3. *Under the assumptions of Theorem 2 for every $\xi\in K$ one has:*

$$\tilde{f}(\xi)=i|\xi|^{-1}\int_{a(\xi^0)}^{b(\xi^0)} dp\exp(ip|\xi|)\int\limits_{S^2\cap\xi^{0\perp}} d\eta\,\mathcal{D}_{\xi^0}\, g(B(p,\xi^0),\eta). \tag{6}$$

Formulas (3) and (6) allow one to calculate $\hat{f}(p,\alpha)$, $p\in\mathbf{R}$, $\alpha\in\omega$, and $\tilde{f}(\xi)$, $\xi\in K$, given the data $g(x,\alpha)$, $x\in L$, $\alpha\in\omega$. We assume that ω is a convex set in S^2. Calculation by formulas (3) and (6) involve two one-dimensional integrations over compact regions.

Given $\hat{f}(p,\alpha)$ for $p\in\mathbf{R}$ and $\alpha\in\omega$, one can calculate $f(x)$ analytically with an arbitrary accuracy. The corresponding formulas are given in Proposition 1 below. They are analogous to the known FBP (filtered back projection) method for complete data ($\omega=S^2$) and are proved in [7].

Pick any function $h(\lambda,\alpha)$, $\lambda\in\mathbf{R}_+$, $\alpha\in S^2$, $\mathbf{R}_+:=(0,\infty)$ with the properties

$$h(\lambda,\alpha)=h(\lambda,-\alpha)=h(-\lambda,\alpha), \tag{7}$$

$$h(\lambda,\alpha)=0, \quad \lambda\alpha\in K, \tag{8}$$

$$\mathcal{F}^{-1}h\big|_{x=0}=1, \tag{9}$$

$$\left|\mathcal{F}^{-1}h\right|\le c_m(1+|x|^2)^{-m} \quad \forall m>0, \quad c_m=\text{const} \tag{10}$$

For example, if $h\in C_0^\infty(K)$, then (10) holds. One can take a function $\eta_1(\lambda)\in C_0^\infty(\mathbf{R}_+)$, $\eta_1\ge0$, then set $\eta_1(-\lambda)=\eta_1(\lambda)$, take $\eta_2(\alpha)\in C_0^\infty(\omega)$, $\eta_2(-\alpha)=\eta_2(\alpha)\ge0$, and let $h(\lambda,\alpha)=c\eta_1(\lambda)\,\eta_2(\alpha)$, where $c=\text{const}>0$ is chosen so that (9) holds. Then Conditions (7)–(10) are met.

Let

$$P_N(|x|^2):=\left(\frac{N}{4\pi\, a^2}\right)^{n/2}\left(1-\frac{|x|^2}{4a^2}\right)^N, \quad n=3, \tag{11}$$

where $a>0$ is a number such that $f(x)=0$ for $|x|>a$. Let Δ be the Laplacian in $\mathbf{R}^3$, $\Delta=\frac{\partial^2}{\partial\lambda^2}+\frac{2}{\lambda}\frac{\partial}{\partial\lambda}+\frac{\Delta^*}{\lambda^2}$, Δ^* is the angular part of Δ. Define

$$w_N(p,\alpha):=\frac{1}{2}(2\pi)^{-n}\int_{-\infty}^{\infty}\exp(i\lambda p)\,|\lambda|^2\,P_N(-\Delta)\,h(\lambda,\alpha)\,d\lambda, \quad n=3. \tag{12}$$

Let

$$w_N * \hat{f} = \int_{-\infty}^{\infty} w_N(p-s,\alpha)\, \hat{f}(s,\alpha) ds. \tag{13}$$

For a function $j(p,\alpha)$ define $R^{\#} j(p,\alpha) = \int_{S^2} j(\alpha \cdot x, \alpha) d\alpha$, and set

$$f_N(x) := R^{\#}(w_N * \hat{f}). \tag{14}$$

Note that $w_N(p,\alpha) = 0$ for $\alpha \notin \omega$, so that Formula (14) uses only the data $\hat{f}(p,\alpha)$ for $\alpha \in \omega$.

PROPOSITION 1 [7]. *If $f \in C^1(\mathbf{R}^3)$ and $f(x) = 0$ for $|x| > a$, then*

$$\sup_{x \in B_a} |f_N(x) - f(x)| \le c N^{-1/2}, \quad N \to \infty, \tag{15}$$

where $c = \text{const} > 0$ depends on $\|f\|_{C^1(B_a)}$.

REMARK. If $f \in L^2_0(B_a)$, then $\|f_N(x) - f(x)\| \to 0$ as $N \to \infty$. Here $L^2_0(B_a)$ is the set of $f \in L^2(B_a)$ with compact support, and B_a is the ball in $\mathbf{R}^3$ with center in the origin of radius a.

We now give formulas for the inversion of the Fourier transform $\tilde{f}(\xi)$ of a function $f \in L^2_0(B_a)$ from the knowledge of $\tilde{f}(\xi)$ on an arbitrary subset $K \in \mathbf{R}^n$ with non-empty interior. In particular, the cone $K = \mathbf{R} \times \omega$ will do. Pick an arbitrary function $\tilde{\chi}(\xi)$:

$$\tilde{\chi}(\xi) \in C_0^\infty(K), \quad \chi(0) = 1, \qquad \text{where } \chi(x) = \mathcal{F}^{-1}\tilde{\chi}(\xi). \tag{16}$$

Define

$$\tilde{\delta}_N(\xi) := P_N(-\Delta_\xi)\, \tilde{\chi}(\xi), \qquad \text{so } \delta_N(x) = P_N(|x|^2)\, \chi(x). \tag{17}$$

Here $\Delta_\xi = \sum_{j=1}^n \dfrac{\partial^2}{\partial \xi_j^2}$ is the Laplacian.

Set

$$f_N(x) := \mathcal{F}^{-1}\left[\tilde{\delta}_N(\xi)\, \tilde{f}(\xi)\right]. \tag{18}$$

Note that $\tilde{\delta}_N(\xi) = 0$ for $\xi \notin K$, so that Formula (18) makes use only of the values of $\tilde{f}(\xi)$ in K.

PROPOSITION 2. *If $f(x) \in L^2_0(B_a)$, then the function (18) has the property*

$$\|f_N(x) - f(x)\|_{L^2(B_a)} \to 0 \qquad \text{as } N \to \infty.$$

REMARK. If $f \in C^1_0(B_a)$, then the function (18) satisfies (15).

Proposition 2 is analogous to the results obtained in [8–11]. Formulas (3), (14) and (6), (18) give a method for the analytical inversion of the incomplete cone beam data with arbitrary accuracy for the exact data. If the data are noisy, then a stable inversion can be obtained by the method described in [9].

In the case $n = 2$, the data $g(x,\beta)$, $\beta \in S^1$, the unit circle, $x \in L$, give directly $\hat{f}(p,\beta^\perp)$. By continuity, a slight rotation of β leads to the direction α for which the basic condition (2) (with α in place of β) holds. Thus, one gets from the cone-beam data the Radon transform $\hat{f}(p,\alpha^\perp)$ data for which Formulas (14) and (15) hold (with $n = 2$).

3. PROOFS

PROOF OF THEOREM 1. If $\hat{f}(p,\alpha)$ is known in K, then $\tilde{f}(\xi)$ is known in K as well (by the known formula $F_{p\to r}\hat{f}(p,\alpha) := \int_{-\infty}^{\infty} \exp(ipr)\, \hat{f}(p,\alpha)\, dp = \tilde{f}(r\alpha)$). Since $\tilde{f}(\xi)$ is an entire function (because $f(x)$ is compactly supported), the knowledge of $\tilde{f}(\xi)$ in K defines $\tilde{f}(\xi)$uniquely in $\mathbf{R}^n$. Thus, $f(x)$ is uniquely defined. The data $g(x,\alpha)$, $x \in L$, $\alpha \in \omega$, determine uniquely $\hat{f}(p,\alpha)$ in K by the formula (3), provided that (2) holds. Thus, Theorem 1 is proved. ∎

PROOF OF THEOREM 2. Put $\alpha = (0,0,1)$, $\eta \cdot \alpha = 0$. Then

$$\mathcal{D}_\alpha g(x,\eta) = \int_0^\infty \frac{\partial f(x+t\eta)}{\partial x_3}\bigg|_{\eta_3=0} t\,dt. \tag{19}$$

Integrate (19) with respect to η over the circle $S^2 \cap \alpha^\perp$ to get

$$\begin{aligned}\int\limits_{S^2\cap\alpha^\perp} \mathcal{D}_\alpha g(x,\eta)\,d\eta &= \int\limits_{S^2\cap\alpha^\perp} d\eta \int_0^\infty \frac{\partial f(x+t\eta)}{\partial x_3}\bigg|_{\eta_3=0} t\,dt \\ &= \int\limits_{\mathbb{R}^2} d\eta_1\,d\eta_2 \,\frac{\partial f(x+(\eta_1,\eta_2,0))}{\partial x_3}.\end{aligned} \tag{20}$$

Take an arbitrary $x \in L$ in (20) such that $\pi_\alpha(x) = \lambda\alpha$, $a(\alpha) < \lambda < b(\alpha)$. This is possible because of the basic assumption (2). Denote this x by $B(\lambda,\alpha)$. Then (20) can be written as

$$\int\limits_{S^2\cap\alpha^\perp} \mathcal{D}_\alpha g(B(\lambda,\alpha),\eta)\,d\eta = \int\limits_{\mathbb{R}^2} \frac{\partial f(\eta_1,\eta_2,\lambda)}{\partial\lambda}\,d\eta_1\,d\eta_2. \tag{21}$$

Note that

$$\int\limits_{\mathbb{R}^2} f(\eta_1,\eta_2,p)\,d\eta_1\,d\eta_2 = \hat{f}(p,\alpha), \tag{22}$$

$\hat{f}(p,\alpha) = 0$ for $p \notin [a(\alpha),b(\alpha)]$. Since $f(x) \in C^1(\mathbf{R}^3)$, the function $\hat{f}(p,\alpha)$ is continuous in p, so $\hat{f}(a(\alpha),\alpha) = 0$. Integrating (21) in λ yields (3). Theorem 2 is proved. ∎

PROOF OF THEOREM 3. Recall that $F_{p\to r}\hat{f}(p,\alpha) = \tilde{f}(r\alpha)$. Multiply (3) by $\exp(ip|\xi|)$ and integrate in p, take into account that $\hat{f}(p,\alpha) = 0$ for $p \le a(\alpha)$ and $p \ge b(\alpha)$, integrate by parts in p and get (6). Theorem 3 is proved. ∎

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